FP3 June 2013 M.A. Kprime2

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1. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{25} = 1, \qquad \text{where } a \text{ is a positive constant.}$$

The foci of H are at the points with coordinates (13, 0) and (–13, 0).

Find

(a) the value of the constant a,
(3)

(b) the equations of the directrices of H.
(3)

1(a). Foci $(ae, 0) \Leftrightarrow (13, 0)$

$\Rightarrow ae = 13$

$\Rightarrow e = \frac{13}{a}$

Eccentricity: $b^2 = a^2(e^2 - 1)$

$\therefore \quad 25 = a^2\left(\frac{169}{a^2} - 1\right)$

$\therefore \quad 25 = 169 - a^2$

$\therefore \quad a^2 = 144 \Rightarrow a = 12$

(b) ~~$x = \pm 12$~~ $\quad e = \frac{13}{12}$

$\therefore \quad x = \pm \frac{144}{13}$

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2. (a) Find

$$\int \frac{1}{\sqrt{(4x^2+9)}}\,dx$$

(2)

(b) Use your answer to part (a) to find the exact value of

$$\int_{-3}^{3} \frac{1}{\sqrt{(4x^2+9)}}\,dx$$

giving your answer in the form $k \ln(a + b\sqrt{5})$, where a and b are integers and k is a constant.

(3)

2(a). $\int \frac{1}{\sqrt{4x^2+9}}\,\partial x = \int \frac{1}{\sqrt{(2x)^2+9}}\,\partial x$

$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + C$

7.253...

~~(b)~~ $\cancel{\int \frac{1}{\sqrt{4(x^2+\frac{9}{4})}}\,\partial x = \frac{1}{2}\int \frac{1}{\sqrt{x^2+9/4}}\,\partial x}$

$\cancel{= \frac{1}{2}\left[= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + C\right.}$

(b) $\left[\frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_{-3}^{3} = \frac{1}{2}\operatorname{arsinh} 2 - \frac{1}{2}\operatorname{arsinh}(-2)$

$= \frac{1}{2}\left(\ln(2+\sqrt{5}) - \ln(-2+\sqrt{5})\right)$

$= \frac{1}{2}\ln(9+4\sqrt{5})$

3. The curve with parametric equations

$$x = \cosh 2\theta, \quad y = 4\sinh\theta, \quad 0 \leqslant \theta \leqslant 1$$

is rotated through 2π radians about the x-axis.

Show that the area of the surface generated is $\lambda(\cosh^3 \alpha - 1)$, where $\alpha = 1$ and λ is a constant to be found.

(7)

3- S. Area $= 2\pi \int_0^1 y \sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2}\, d\theta$

$x = \cosh 2\theta \Rightarrow \frac{\partial x}{\partial\theta} = 2\sinh 2\theta$

$y = 4\sinh\theta \Rightarrow \frac{\partial y}{\partial\theta} = 4\cosh\theta$

$(\sinh 2\theta)^2 = 4\sinh^2\theta\cosh^2\theta$

$\therefore$ Area $= 2\pi \int_0^1 4\sinh\theta \sqrt{4\sinh^2 2\theta + 16\cosh^2\theta}\, \partial\theta$

$c^2 - s^2 = 1$

$\cosh^2\theta = 1 + \sinh^2\theta$

$= 2\pi \int_0^1 4\sinh\theta \sqrt{16\sinh^2\theta\cosh^2\theta + 16\cosh^2\theta}\, \partial\theta$

$= 2\pi \int_0^1 4\sinh\theta \sqrt{16\cosh^2\theta(\sinh^2\theta + 1)}\, \partial\theta$

$= 2\pi \int_0^1 4\sinh\theta \sqrt{16\cosh^4\theta}\, \partial\theta$

Question 3 continued

$$= 8\pi \int_0^1 \sinh\theta \cdot 4\cosh^2\theta \; d\theta$$

$$= 32\pi \int_0^1 \sinh\theta \cosh^2\theta \; d\theta \qquad \int f'(x)[f(x)]^n dx$$

$$= 32\pi \left[\frac{\cosh^3\theta}{3} \right]_0^1$$

$$= 32\pi \left(\frac{\cosh^3(1)}{3} - \frac{\cosh^3(0)}{3} \right)$$

$$= 32\pi \left(\frac{1}{3}\cosh^3(1) - \frac{1}{3} \right)$$

$$= \frac{32}{3}\pi \left(\cosh^3(1) - 1 \right)$$

$$\lambda = \frac{32}{3}\pi$$

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4.

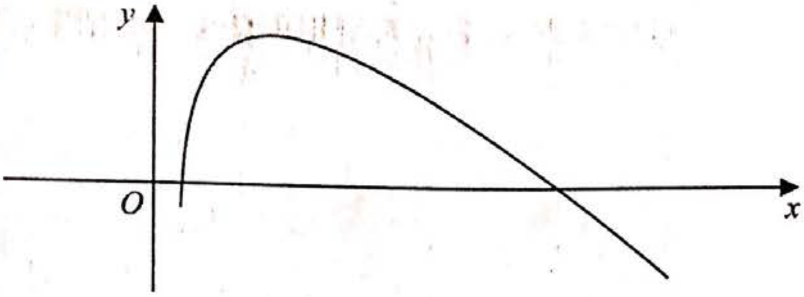

Figure 1

Figure 1 shows part of the curve with equation

$$y = 40 \operatorname{arcosh} x - 9x, \qquad x \geqslant 1$$

Use calculus to find the exact coordinates of the turning point of the curve, giving your answer in the form $\left(\frac{p}{q}, r\ln 3 + s\right)$, where p, q, r and s are integers.

(7)

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4. $y = 40 \operatorname{arcosh} x - 9x$

$\therefore \frac{dy}{dx} = \frac{40}{\sqrt{x^2-1}} - 9$

$\frac{dy}{dx} = 0 \Rightarrow \frac{40}{\sqrt{x^2-1}} - 9 = 0$

$\therefore 40 = 9\sqrt{x^2-1}$

$\Rightarrow 40^2 = \left(9\sqrt{x^2-1}\right)^2$

$\therefore 1600 = 81(x^2-1)$

$\therefore \frac{1600}{81} = x^2 - 1 \Rightarrow x = + \frac{41}{9}$

Question 4 continued

$x = \frac{41}{9}$

$\Rightarrow \quad y = 40\,\text{arcosh}\,\frac{41}{9} - 41$

$= 40\left(\ln\left(\frac{41}{9} + \sqrt{\frac{1600}{81}}\right)\right) - 41$

$= 40\ln\left(\frac{41}{9} + \frac{40}{9}\right) - 41$

$= 40\ln(9) - 41$

$= 40\ln(3^2) - 41$

$= 80\ln(3) - 41$

$\therefore$ Turning point: $\left(\frac{41}{9}, 80\ln(3) - 41\right)$

$p = 41 \quad q = 9 \quad r = 80$
$s = -41$

Q4

(Total 7 marks)

5. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix}, \text{ where } a, b \text{ and } c \text{ are constants.}$$

(a) Given that $\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{k}$ are two of the eigenvectors of $\mathbf{M}$,

find

(i) the values of a, b and c,

(ii) the eigenvalues which correspond to the two given eigenvectors.

(8)

(b) The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & d \\ -1 & 0 & 1 \end{pmatrix}, \text{ where } d \text{ is constant, } d \neq -1$$

Find

(i) the determinant of $\mathbf{P}$ in terms of d,

(ii) the matrix $\mathbf{P}^{-1}$ in terms of d.

(5)

5(a) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are E.vectors

$M x = \lambda x \Rightarrow \begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ \lambda \end{pmatrix}$

$\therefore \begin{pmatrix} 1+a \\ b+c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ \lambda \end{pmatrix} \Rightarrow$ $a = -1$, $\lambda = 1$, $b + c = 1$

Question 5 continued

$$\& \begin{pmatrix} 1 & 1 & -1 \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ -\lambda \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 \\ 2-c \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ -\lambda \end{pmatrix} \Rightarrow \begin{array}{l} \lambda = 2 \\ c = 2 \end{array}$$

$$\therefore \quad b+2=1 \Rightarrow b=-1$$

(i) $a=-1$
$b=-1$
$c=2$

(ii) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ has e. value $\lambda = 1$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ has e. value $\lambda = 2$

(b)(i) $\det(P) = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 2 & d \\ -1 & 1 \end{vmatrix} + 0$

$= 1 - (2+d) = 1-2-d$

$= -(1+d)$

Question 5 continued

(ii)

$$C = \begin{pmatrix} +(1) & -(2+d) & +(1) \\ -(1) & +(1) & -(1) \\ +(d) & -(d) & +(-1) \end{pmatrix}$$

$$\therefore C^T = \begin{pmatrix} 1 & -1 & d \\ -2-d & 1 & -d \\ 1 & -1 & -1 \end{pmatrix}$$

$$\therefore P^{-1} = \frac{-1}{d+1} \begin{pmatrix} 1 & -1 & d \\ -2-d & 1 & -d \\ 1 & -1 & -1 \end{pmatrix}$$

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6. Given that

$$I_n = \int_0^4 x^n \sqrt{(16 - x^2)}\,dx, \quad n \geqslant 0,$$

(a) prove that, for $n \geqslant 2$,

$$(n+2)I_n = 16(n-1)I_{n-2}$$

(6)

(b) Hence, showing each step of your working, find the exact value of I_5

(5)

(a). $I_n = \int_0^4 x^n \sqrt{16-x^2}\ \partial x$

$= -\frac{1}{2} \int_0^4 x^{n-1} \cdot -2x \sqrt{16-x^2}\ \partial x$

Let $u = x^{n-1}$ $\quad u' = (n-1)x^{n-2}$

$v' = -2x\sqrt{16-x^2} \Rightarrow \quad v = \frac{2}{3}(16-x^2)^{3/2}$

$\therefore -2I_n = \left[\frac{2x^{n-1}}{3}(16-x^2)^{3/2}\right]_0^4 - \frac{2}{3}(n-1)\int_0^4 x^{n-2}(16-x^2)^{3/2}\,\partial x$

$\therefore -2I_n = -\frac{2}{3}(n-1)\int_0^4 x^{n-2}\left(\sqrt{16-x^2} \cdot 16-x^2\right)\partial x$

$\therefore \quad I_n = \frac{n-1}{3}\int_0^4 16x^{n-2}\sqrt{16-x^2} - x^n\sqrt{16-x^2}\ \partial x$

$$\therefore I_n = \frac{n-1}{3}\left(16\int_0^4 x^{n-2}\sqrt{16-x^2}\,dx - \int_0^4 x^n\sqrt{16-x^2}\,dx\right)$$

$$\therefore I_n = \frac{n-1}{3}\left(16 I_{n-2} - I_n\right)$$

$$\therefore I_n = \frac{16}{3}(n-1) I_{n-2} - \frac{n-1}{3} I_n$$

$$\therefore \left(1 + \frac{n-1}{3}\right) I_n = \frac{16}{3}(n-1) I_{n-2}$$

$$\therefore (n+2) I_n = 16(n-1) I_{n-2}$$

as required.

Question 6 continued $(n+2) I_n = 16(n-1) I_{n-2}$

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(b)

let $n=5$

$\Rightarrow \quad 7 I_5 = 64 I_3$

now $n=3$

$\Rightarrow \quad 5 I_3 = 32 I_1 \Rightarrow I_3 = \frac{32}{5} I_1$

$\therefore \quad 7 I_5 = \frac{2048}{5} I_1$

$\Rightarrow \quad I_5 = \frac{2048}{35} I_1$

$\Rightarrow I_5 = \frac{2048}{35} \int_0^4 x\sqrt{16-x^2}\, dx$

$= \frac{2048}{35} \cdot -\frac{1}{2} \int_0^4 -2x \left(16-x^2\right)^{1/2} dx$

$= -\frac{1024}{35} \left[\frac{2\left(16-x^2\right)^{3/2}}{3} \right]_0^4$

$= -\frac{1024}{35} \left(0 - \frac{128}{3} \right) = \frac{1024}{35} \times \frac{128}{3} = \frac{131072}{105}$

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7. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0$$

The line l is a normal to E at a point $P(a\cos\theta, b\sin\theta)$, $0 < \theta < \frac{\pi}{2}$

(a) Using calculus, show that an equation for l is

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

(5)

The line l meets the x-axis at A and the y-axis at B.

(b) Show that the area of the triangle OAB, where O is the origin, may be written as $k\sin 2\theta$, giving the value of the constant k in terms of a and b.

(4)

(c) Find, in terms of a and b, the exact coordinates of the point P, for which the area of the triangle OAB is a maximum.

(3)

7(a). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{\partial y}{\partial x} = \frac{\partial y/\partial\theta}{\partial x/\partial\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

$\therefore \; m_N = -\frac{1}{-\frac{b}{a}\frac{\cos\theta}{\sin\theta}} = \frac{a}{b}\tan\theta$

$\therefore \; y - b\sin\theta = \frac{a}{b}\tan\theta(x - a\cos\theta)$

$\therefore \; y - b\sin\theta = \frac{a}{b}\frac{\sin\theta}{\cos\theta}x - \frac{a^2}{b}\sin\theta$

$\therefore \; by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$\therefore \; ax\sin\theta - by\cos\theta = a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta$

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Question 7 continued

$\therefore \ ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

as required.

(b) at A, $y=0 \Rightarrow \ ax\sin\theta = (a^2-b^2)\sin\theta\cos\theta$

$\Rightarrow \ x = \dfrac{a^2-b^2}{a}\cos\theta$

~~$A\left(\dfrac{a^2-b^2}{a}\cos\theta, 0\right)$~~

$A\left(\dfrac{a^2-b^2}{a}\cos\theta \ , \ 0\right)$

At B, $x=0 \Rightarrow \ -by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

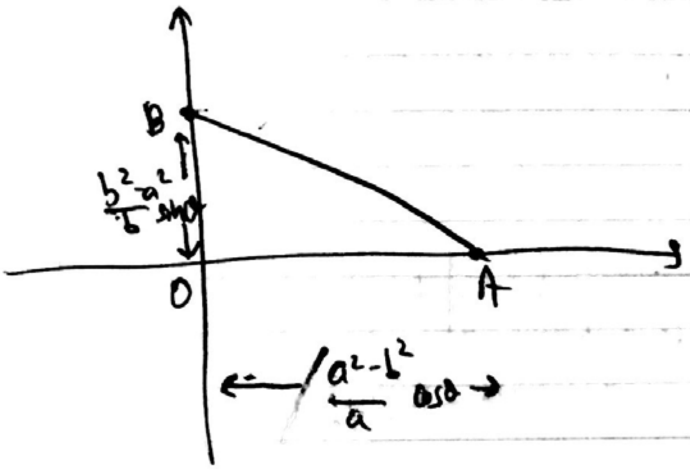
$\downarrow \div -b\cos\theta$

$\therefore$

$y = \dfrac{b^2-a^2}{b}\sin\theta$

$\therefore B\left(0, \ \dfrac{b^2-a^2}{b}\sin\theta\right)$

Question 7 continued



$\sin\theta\cos\theta \equiv \frac{1}{2}\sin 2\theta$

$$\therefore \text{Area} = \left| \frac{1}{2} \times \frac{a^2-b^2}{a}\cos\theta \times \frac{b^2-a^2}{b}\sin\theta \right|$$

$$= \left| \frac{1}{2} \cdot \frac{(a^2-b^2)(b^2-a^2)}{ab} \cdot \frac{1}{2}\sin 2\theta \right|$$

$$= \left| \frac{(a^2-b^2)(b^2-a^2)}{4ab} \sin 2\theta \right|$$

$$= \left| -\frac{(a^2-b^2)^2}{4ab} \sin 2\theta \right|$$

$$\text{Area} = \frac{(a^2-b^2)^2}{4ab}\sin 2\theta$$

$$k = \frac{+(a^2-b^2)^2}{4ab}$$

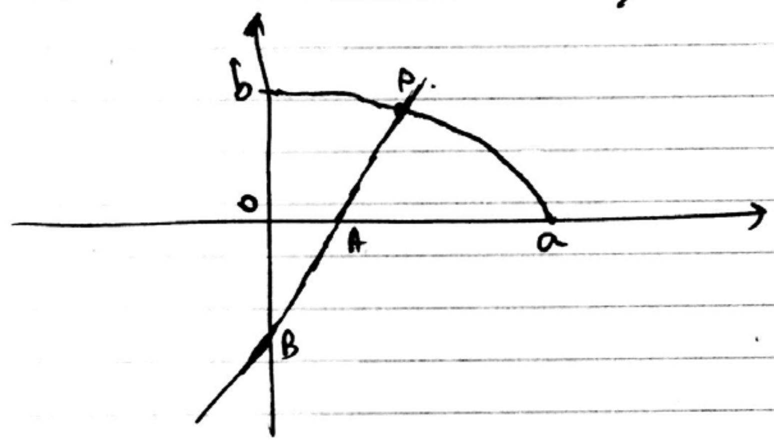
Question 7 continued

(c) ~~Ar~~ let Area $= A$

$$A = \frac{(a^2-b^2)^2}{4ab} \sin 2\theta$$

Area is maximum when $\sin 2\theta = 1$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$



$\therefore$ Area is max for $\theta = \frac{\pi}{4}$

$$\rightarrow P\left(a\cos\frac{\pi}{4},\quad b\sin\frac{\pi}{4}\right)$$

$$\therefore \quad P: \left(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right)$$

Q7

(Total 12 marks)

Turn over

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8. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k}), \text{ where } \lambda \text{ and } \mu \text{ are scalar parameters.}$$

(b) Find the acute angle between Π_1 and Π_2 giving your answer to the nearest degree.

(5)

(c) Find an equation of the line of intersection of the two planes in the form $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

(6)

8(a). $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 5$

$\Rightarrow \quad 3x - 4y + 2z = 5$

$\Rightarrow \quad 3x - 4y + 2z - 5 = 0$

$\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \begin{pmatrix} 6 \\ 2 \\ 12 \end{pmatrix}$

Use formula in booklet pg 10.

$$d = \frac{\left| 3(6) - 4(2) + 2(12) - 5 \right|}{\sqrt{3^2 + 4^2 + 2^2}}$$

$$= \frac{29}{\sqrt{29}} = \sqrt{29}$$

(b) $\underline{\Pi_2:}$

$$\underset{\sim}{n} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$= \quad \begin{array}{cccccc} 2 & 1 & 5 & 2 & 1 & 5 \\ & & \times \quad \times & \times & & \\ 1 & -1 & -2 & 1 & -1 & -2 \end{array}$$

$$= \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix}$$

$$\therefore \cos\theta = \left| \frac{\begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}}{\sqrt{3^2+9^2+3^2}\ \sqrt{3^2+4^2+2^2}} \right|$$

$$= \left| \frac{-33}{3\sqrt{319}} \right| = \left| -\frac{\sqrt{319}}{29} \right|$$

$$\therefore \cos\theta = \frac{\sqrt{319}}{29} \Rightarrow \theta = 52^\circ \text{ (nearest degree)}$$

Question 8 continued

(c) Π_2 : $\cancel{\#}$ $\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix}$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} = 0$

$\therefore \quad 3x + 9y - 3z = 0$

$\therefore \quad x + 3y - z = 0 \quad \Rightarrow \quad x = z - 3y$

& $\Pi_1 \Rightarrow \quad 3x - 4y + 2z = 5$

$\therefore \quad 3z - 9y - 4y + 2z = 5$

$\therefore \quad 5z - 13y = 5$

$\Rightarrow \quad 5z = 5 + 13y$

$\therefore z = 1 + \frac{13}{5}y$

~~$\therefore x = z - 3\left(1 + \frac{13}{5}y\right) = z$~~

$x = 1 + \frac{13}{5}y - 3y = 1 - \frac{2}{5}y$

$y = y$

$z = 1 + \frac{13}{5}y$

Leave blank

Question 8 continued

$\Rightarrow$ line of intersection has eqn

Let ~~y = λ~~ $y = t$

$$\underline{r} = \begin{pmatrix} 1 - \frac{2}{5}t \\ t \\ 1 + \frac{13}{5}t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix}$$

$$(\underline{r} - \underline{a}) \times \underline{b} = 0$$

$$\therefore \quad \underline{r} \times \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix}$$

$$\begin{matrix} 1 & 0 & 1 & & 1 & 0 & 1 \\ -\frac{2}{5} & 1 & \frac{13}{5} & & -\frac{2}{5} & 1 & \frac{13}{5} \end{matrix}$$

$$\begin{pmatrix} -1 \\ \cancel{-3} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$r \times \frac{1}{5}\begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$\therefore r \times \begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} -5 \\ -15 \\ 5 \end{pmatrix}$$